

Antimagic Behavior of SG_n^p and its Subdivision

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Abstract

A finite simple graph G with a subgraph H is called a super (b, d) - H -antimagic: if G has an edge covering by subgraphs $H_1, H_2, \dots, H_t$ with each $H_i \cong H$, $i = 1, 2, \dots, t$, a total labeling α such that $wt_\alpha H$, constitutes an arithmetic progression and $\alpha(V(G))$ consists of the smallest possible integers. In this manuscript, we investigated the existence of super $(b, 1)$ -star-antimagic labeling of Sun graphs SG_n^p .

Keywords: Star graph S_n , Sun graph SG_n^p , super $(b, 1)$ - S_{p+2} -antimagic.

1. Introduction

If each edge of a graph G belongs to one of the subgraphs $H_1, H_2, \dots, H_t$ with $H_i \cong H$, H_i , $i = 1, 2, \dots, t$ then G admits an H -covering. A graph G with an H -covering is (b, d) - H -antimagic if for a total labeling $\alpha : V(G) \cup E(G) \rightarrow \{1, 2, \dots, |V(G)| + |E(G)|\}$, the H -weights,

$$wt_\alpha(H) = \sum_{x \in V(H)} \alpha(x) + \sum_{y \in E(H)} \alpha(y)$$

constitute an arithmetic progression $\{b, b + d, \dots, b + (t - 1)d\}$, where $b > 0$ and $d \geq 0$ are two integers and t is the number of all subgraphs $H_i \cong H$. Furthermore, α is super (b, d) - H -antimagic labeling if $\alpha(V(G)) = \{1, 2, \dots, |V|\}$.

The (super) H -magic graph was first introduced by Gutiérrez and Lladó in [4]. Proved results are about: star $K_{1,n}$ graphs, complete bipartite graphs $K_{n,m}$, paths P_n , cycles C_n , wheels, windmills, books, prisms, shrubs and banana tree graphs. Lladó and Moragas [9] investigated C_n -(super)magic graphs and proved that wheels, windmills, books and prisms are C_h -magic for some h . Some results on C_n -supermagic labelings of several classes of graphs can be found in [13]. Maryati et al. [10] gave P_h -(super)magic labelings of shrubs, subdivision of shrubs and banana tree graphs. Other examples of H -supermagic graphs with different choices of H have been given by Jeyanthi and Selvagopal in [8]. Maryati et al. [11] investigated the G -supermagicness of a disjoint union of c copies of a graph G and showed that disjoint union of any paths is cP_h -supermagic for some c and h . The (b, d) - H -antimagic labeling was introduced by Inayah et al. [6]. Inayah et al. in [6], [7] introduced (b, d) - H -antimagic labeling for some shackles of a connected graph H .

The present paper investigated the super $(b, 1)$ - S_{p+2} -antimagic labeling of Sun graphs SG_n^p with p pendant edges.

2. Star antimagicness of Sun graphs

A star graph S_n is a tree consisting of one vertex adjacent to n vertices. In other words, a complete bipartite graph $K_{1,n}$ is called a Star S_n .

The Sun graph SG_n^p , $n \geq 3, p \geq 1$, is constructed from a cycle C_n by inserting p pendant edges with every vertex of the C_n . The vertices and edges on the cycle will be called the cycle vertices and the cycle edges respectively. The remaining edges will be termed as the pendant edges and their end points as pendant vertices. The Sun Graph SG_n^p contains $n(p + 1)$ vertices and edges.

$$V(SG_n^p) = \{x_i, y_i^j : 1 \leq j \leq p\}, \text{ and}$$

$$E(SG_n^p) = \{x_i x_{i+1}, x_i y_i^j : 1 \leq j \leq p\},$$

where indices i are taken modulo n .

2.1 Super S_{p+2} -antimagic Labeling of Sun Graph SG_n^p

Let S_{p+2} be a star on $p+3$ vertices. Every star $S_{p+2}^k, k = 1, 2, \dots, n$ in SG_n^p has the vertex set

$$V(S_{p+2}^{(k)}) = \{x_{k-1}, x_k, x_{k+1}\} \cup \{y_k^j : 1 \leq j \leq p\}$$

and the edge set

$$E(S_{p+2}^{(k)}) = \{x_{k-1}x_k, x_kx_{k+1}, x_ky_k^j : 1 \leq j \leq p\},$$

where indices are taken modulo n .

Under a total labeling α , the $S_{p+2}^{(k)}$ -weights are:

$$\begin{aligned} wt_\alpha(S_{p+2}^{(k)}) &= \sum_{v \in V(S_{p+2}^{(k)})} \alpha(v) + \sum_{e \in E(S_{p+2}^{(k)})} \alpha(e). \\ &= \sum_{s=k-1}^k (\alpha(x_s) + \alpha(x_sx_{s+1})) + \alpha(x_{k+1}) + \sum_{s=1}^p (\alpha(y_k^s) + \alpha(y_k^s x_k)) \\ &= Sum_1 + Sum_2 \end{aligned} \tag{1}$$

where

$$Sum_1 = \sum_{s=k-1}^k (\alpha(x_s) + \alpha(x_sx_{s+1})) + \alpha(x_{k+1}) \tag{2}$$

$$Sum_2 = \sum_{s=1}^p (\alpha(y_k^s) + \alpha(y_k^s x_k)) \tag{3}$$

Theorem 2.1. The Sun graph SG_n^p admits a super $(b, 1)$ - S_{p+2} -antimagic labeling where $n \geq 3, p \geq 1$ are positive integers and S_{p+2} is a star on $p+3$ vertices.

Proof. Proof is divided into two parts:

In the first part, we give a P_3 -antimagic labeling of cycle C_n . In second part, we show that weights of pendant edges with P_3 -antimagic labeling of cycle C_n gives us super $(b, 1)$ - S_{p+2} -antimagic labeling.

The total labeling α for cycle C_n is defined as:

$$\begin{aligned} \alpha(x_i) &= i \\ \alpha(x_i x_{i+1}) &= 2n(p+1) + 1 - i \end{aligned}$$

where indices are taken modulo n . Under labeling α , cycle vertices are labeled with $\{1, 2, \dots, n\}$ and cycle edges are labeled with $\{n(2p+1) + 1, 2, \dots, 2n(p+1)\}$.

Using (2) and above labeling, the sum for $P_3^{(k)}$ -weights are:

$$\begin{aligned} Sum_1 &= \alpha(x_{k-1}) + \alpha(x_k) + \alpha(x_{k+1}) + \alpha(x_{k-1}x_k) + \alpha(x_kx_{k+1}) \\ &= 3k + 4n(p+1) + 3 - 2k \\ &= 4n(p+1) + 3 + k \end{aligned} \tag{4}$$

which shows a super P_3 -antimagic labeling of cycle C_n .

For $j = 1, 2, \dots, p$ and $j \equiv 1 \pmod{2}$, the labeling of pendent edges and their end vertices is defined as:

$$\{\alpha(y_i^j), \alpha(x_i y_i^j)\} = \{(j+1)n + 1 - i, (p+j)n + i\}$$

For $j = 1, 2, \dots, p, j \equiv 0 \pmod{2}$

$$\{\alpha(y_i^j), \alpha(x_i y_i^j)\} = \{jn + i, (p+j+1)n + 1 - i\}$$

where the smallest possible labels $\{n+1, n+2, \dots, n(p+1)\}$ appear on the end vertices of pendant edges and the pendant edges receive labels $\{n(p+1) + 1, n(p+1) + 1 + 2, \dots, n(2p+1)\}$. Therefore α is total labeling of SG_n^p .

Also, for $j = 1, 2, \dots, p$

$$\alpha(y_i^j) + \alpha(x_i y_i^j) = n(p+2j+1) + 1 \tag{5}$$

Using (3) and (5), we have

$$Sum_2 = \sum_{s=1}^p (\alpha(y_{(k,s)}) + \alpha(x_k y_{(k,s)})) = p[2n(p+1) + 1] \quad (6)$$

Equation (1), (4) and (6) gives

$$\begin{aligned} wt_\alpha(S_{p+2}^{(k)}) &= 4n(p+1) + 3 + k + p[2n(p+1) + 1] \\ &= 2n(p+1)(p+2) + p + 3 + k \end{aligned}$$

which consists of consecutive integers with initial term $a = 2n(p+1)(p+2) + p + 4$. This completes the proof. $\square$

2.2 $S_3(r)$ antimagic labeling of r -Subdivided Sun graph $SG_n^{(1)}(r)$

Let $G(r)$ be a graph (denoted as r -subdivided graph of G) obtained from the graph G by inserting $r \geq 1$ new vertices into every edge of G .

Let $SG_n^1(r)$ be r -subdivided graph of Sun graph SG_n^1 with $2n(r+1)$ vertices and edges.

$$V(SG_n^1(r)) := \{x_{(i,j)}, y_{(i,j)} : 1 \leq i \leq n, 0 \leq j \leq r\}$$

$$E(SG_n^1(r)) := \{x_{(i,j)}x_{(i,j+1)}, y_{(i,j)}y_{(i,j+1)} : 0 \leq j \leq r-1\} \cup \{x_{(i,r)}x_{(i+1,0)}, y_{(i,r)}y_{(i,0)}\}$$

with indices are taken modulo n .

Let $S := S_3(r)$ be r -subdivided graph of star graph S_3 with $3(r+1) + 1$ vertices and $3(r+1)$ edges.

The k^{th} r -subdivided star $S^{(k)}$ has the vertex set

$$V(S^{(k)}) = \{x_{(k-1,0)}, x_{(k,0)}, x_{(k+1,0)}\}, \{y_{(k,j)} : 0 \leq j \leq r\}$$

and the edge set as follows:

$E(S^{(k)}) = \{x_{(k-1,j)}x_{(k-1,j+1)}, x_{(k,j)}x_{(k,j+1)}, y_{(k,j)}y_{(k,j+1)} : 0 \leq j \leq r-1\} \cup \{y_{(k,r)}x_{(k,0)}, x_{(k,r)}x_{(k+1,0)}\}$, where indices are taken modulo n .

Under a total labeling β , the $wt_\beta(S) := wt(S_3(r))$ are:

$$\begin{aligned} wt_\beta(S^{(k)}) &= \sum_{v \in V(S^{(k)})} \beta(v) + \sum_{e \in E(S^{(k)})} \beta(e) \\ &= \sum_{s=k-1}^{k+1} \beta(x_{(s,0)}) + \sum_{s=k-1}^k \sum_{j=0}^{r-1} (\beta(x_{(s,j)})\beta(x_{(s,j+1)})) + \sum_{s=k-1}^k \beta(x_{(s,r)}x_{(s+1,0)}) \\ &\quad + \sum_{j=0}^r \beta(y_{(k,j)}) + \sum_{j=0}^{r-1} \beta(y_{(k,j)})\beta(y_{(k,j+1)}) + \beta(y_{(k,r)}x_{(k,0)}) \\ &= Sum_1 + Sum_2 \end{aligned} \quad (7)$$

where

$$Sum_1 = \sum_{s=k-1}^{k+1} \beta(x_{(s,0)}) + \sum_{s=k-1}^k \sum_{j=0}^{r-1} (\beta(x_{(s,j)})\beta(x_{(s,j+1)})) + \sum_{s=k-1}^k \beta(x_{(s,r)}x_{(s+1,0)}) \quad (8)$$

$$Sum_2 = \sum_{j=0}^r \beta(y_{(k,j)}) + \sum_{j=0}^{r-1} \beta(y_{(k,j)})\beta(y_{(k,j+1)}) + \beta(y_{(k,r)}x_{(k,0)}) \quad (9)$$

Theorem 2.2. The Sun graph $SG_n(r)$ admits a super $(b,1)$ - S -antimagic labeling where $n \geq 3$, $r \geq 1$ are positive integers and S is an r -subdivided graph of S_3 .

Proof. Firstly, we give a $P_3(r)$ -antimagic labeling of r -subdivide cycle $C_n(r)$. The total labeling β for r -subdivided cycle $C_n(r)$ is defined as:

$$\begin{aligned} \beta(x_{(i,j)}) &= nj + i & j &= 0, 1, 2, \dots, r \\ \beta(x_{(i,j)}x_{(i,j+1)}) &= n(4r + 4 - j) + 1 - i & j &= 0, 1, 2, \dots, r-1 \\ \beta(x_{(i,r)}x_{(i+1,0)}) &= n(4r + 4 - r) + 1 - i & j &= 0, 1, 2, \dots, r-1 \end{aligned}$$

where indices are taken modulo n . Under labeling β , vertices of r -subdivided cycle are labeled with $\{1, 2, \dots, n(r+1)\}$ and edges of r -subdivided cycle are labeled with $\{3n(r+1) + 1, 3n(r+1) + 2, \dots, 4n(r+1)\}$.

Using (8) and above labeling, the sum for $P_3^{(k)}(r)$ -weights are:

$$\sum_{s=k-1}^{k+1} \beta(x_{(s,0)}) + \sum_{s=k-1}^k \sum_{j=0}^{r-1} (\beta(x_{(s,j)})\beta(x_{(s,j+1)})) + \sum_{s=k-1}^k \beta(x_{(s,r)}x_{(s+1,0)}) = (r+1)\{4n(r+1) + 1\} + i$$

$$= (r+1)(4nr + 4n + 1) + i \quad (10)$$

which shows a super $P_3(r)$ -antimagic labeling of r -subdivided cycle $C_n(r)$.

Labeling of pendent edges and their end points is defined as:

$$\begin{aligned} \beta(y_i^j) &= n(2r + 2 - j) + 1 - i & j &= 0, 1, 2, \dots, r \\ \beta(y_i^j y_{(i,j+1)}) &= n(2r + 2 + j) + i & j &= 0, 1, 2, \dots, r-1 \\ \beta(y_{(i,r)} x_{(i,0)}) &= n(2r + 2 - r) + i & j &= 0, 1, 2, \dots, r-1 \end{aligned}$$

where the smallest possible labels $\{n(r+1) + 1, n(r+1) + 2, \dots, 2n(r+1)\}$ appear on the end points of pendant edges and the pendant edges receive labels $\{2n(r+1) + 1, 2n(r+1) + 2, \dots, 3n(r+1)\}$. Therefore β is total labeling of SG_n^p .

$$\sum_{j=0}^r \beta(y_{(k,j)}) + \sum_{j=0}^{r-1} \beta(y_{(k,j)})\beta(y_{(k,j+1)}) + \beta(y_{(k,r)}x_{(k,0)}) = (r+1)\{4n(r+1) + 1\}$$

$$= 4n(r+1)^2 + (r+1) \quad (11)$$

Using equations (7), (10) and (11), we get

$$\begin{aligned} wt_\beta(S^{(k)}) &= 8n(r+1)^2 + 2(r+1) + i \\ &= 2(r+1) + (4nr + 4n + 1) + i \end{aligned}$$

which constitute a sequence of consecutive integers with initial term

$a = 8n(r+1)^2 + 2(r+1) + i$. This completes the proof. □

3. Conflict of Interests

The author(s) declare that there is no conflict of interests.

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