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Normalized Cauchy Projections and Boundary K-Homology for Higher-Order Elliptic Operators

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Abstract

The higher-order elliptic differential operator on a compact manifold with boundary defines a relative analytic K-homology class through the minimal realization, whereas the maximal realization has the boundary contribution. In this paper, we construct normal-trace compression of the boundary class of an operator

$$D : C^\infty(\Omega; E) \longrightarrow C^\infty(\Omega; F)$$

of order m. We start with the even operator

$$\mathcal{D}_D = \begin{pmatrix} 0 & (D^\dagger)_{\min} \\ D_{\min} & 0 \end{pmatrix},$$

where the generalized Bergman space is the kernel $\operatorname{Ker} D_{\max}$, take the complete trace vector $\gamma u = (\gamma_0 u, \ldots, \gamma_{m-1} u)$, and match the mixed Sobolev trace levels using the diagonal elliptic operator

$$\lambda_{1/2} = \operatorname{diag}(\Lambda_{1/2}, \Lambda_{3/2}, \ldots, \Lambda_{m-1/2}).$$

The normalized Cauchy range

$$\widehat{H}_D = \lambda_{1/2}^{-1} \gamma(\operatorname{Ker} D_{\max}) \subset L^2(\partial\Omega; E \otimes \mathbb{C}^m)$$

has the orthogonal projection Π_D, which defines an odd Fredholm module over $C(\partial\Omega)$. We obtain the following identity:

$$\partial[\mathcal{D}_D] = [\Pi_D] = \varphi_*\big([E_+(D)] \cap [S^*\partial\Omega]\big).$$

It shows that normal trace compression includes precisely the K-homology boundary operator's image and the Calderón symbol. The analysis also establishes that admissible extensions do not alter the interior class, low order perturbations retain the boundary K-class provided that the highest boundary symbol remains intact, and formal self-adjointness implies vanishing of scalar Toeplitz pairing while preserving the Cauchy data that is not identically zero. The answer is thus specific: the higher-order boundary class is the Fredholm module associated with the normalized full Cauchy projection.

Keywords: higher-order elliptic operators; normal traces; Calderon projection; relative K-homology; boundary map; generalized Bergman space; Fredholm modules; elliptic boundary value theory

1. Boundary K-Homology and the Need for Full Trace Compression

The Atiyah–Singer theorem related the Fredholm index of an elliptic operator on a closed manifold to topological symbol data [1]. The full theorem stated the index formula in a theory of elliptic operators [2]. Seminars clarified the analytic and topological elements of the theorem [3]. When a manifold has a boundary, it is not sufficient for the interior symbol to define a Fredholm realization. Parameter-dependent elliptic theory showed that boundary conditions and normal roots played a role in the solvability problem [4]. The topological obstruction was present already in the Atiyah–Bott index theorem for manifolds with boundary [5]. In the first Atiyah–Patodi–Singer paper, a spectral boundary condition defined by the tangential operator was considered [6]. The second paper elaborated on the spectral-asymmetry framework [7]. The third paper completed the series and its geometric applications [8]. Later, Melrose reformulated the index theorem within boundary calculus [9]. Boutet de Monvel's calculus provides a pseudodifferential framework for boundary operators and traces [10]. The Calderon boundary construction first singled out the admissible Cauchy data of an elliptic equation [11]. Seeley's projection identified those boundary values via a pseudodifferential idempotent [12].

Analytic K-homology furnishes the homological language for this obstruction. The global theory of Atiyah associates elliptic operators with index homological data [13]. The operator K-functor of Kasparov serves as the framework for extension of bivariance [14]. Baum and Douglas formulated K-homology geometrically using cycles and index theory [15]. Relative cycles give the description of the passage from an interior ideal to a boundary quotient [16]. Analytic K-homology constructs the same classes using Fredholm modules and bounded transforms [17]. Extension theory gave the operator-algebraic approach to constructing K-homology from C*-algebra extensions [18]. Standard K-theory texts treat these constructions within the algebraic framework [19]. The spectral triples of Connes provide a formulation of first-order non-commutative geometry via self-adjoint unbounded operators satisfying the bounded commutation condition [20]. Higher-order differential operators need another regularity condition. If D has the order m, then the multiplication by a smooth function gives

$$\operatorname{ord}[D, a] \leq m - 1.$$

The correct boundedness condition is thus

$$[D, a](1 + D^*D)^{-1/2+1/(2m)} \in \mathcal{B}(\mathcal{H}),$$

and compensates for one missing derivative. Boundary spectral triples replace the self-adjointness by the symmetry and encode the relative ideal explicitly [21]. The work on finite-summability shows how regularity conditions affect the construction of spectral triples in the non-classical manifold case [22]. Higher-order relative K-homology uses the interior condition $j \operatorname{Dom} D^* \subseteq \operatorname{Dom} D$ for the functions in the interior ideal [23].

Unbounded representatives are necessary here. Unbounded operators representing bivariant K-classes before the transformation to bounded ones were shown by Baaj and Julg [24]. The conditions for the Kasparov product of unbounded modules were provided by Kucerovsky [25]. Mesland gave the unbounded bivariant K-theory via correspondences in non-commutative geometry [26]. The local–global principle for regular operators in Hilbert C*-modules was proved by Kaad and Lesch [27]. The reason for this is the necessity of the non-classical regularity of the higher-order relative operator. The properties of the graph domain, adjoint domain, compactness estimates, and regularized commutators decide whether the bounded transform gives a K-cycle.

The classical trace theory explains why a higher-order boundary class cannot be represented by a scalar trace. The general partial differential equation theory gives the Sobolev and trace theory for differential operators [28]. For an operator of order m, the boundary vector contains m components, each of which is in a different Sobolev order. The Lions–Magenes hierarchy describes this situation for non-homogeneous boundary value problems [29]. McLean gives the treatment for strongly elliptic systems and boundary integral equations [30]. Pseudodifferential calculus gives the symbol and mapping properties for these trace operators [31]. The spectral analysis of pseudodifferential operators gives the functional analytic treatment [32]. The Calderon idempotent applied to these data belongs to the Douglis–Nirenberg calculus, but not to the scalar single-order calculus. Normal-trace compression retains all trace components and matches their orders only after the complete Cauchy space is constructed.

The question answered by this article is the following: given an elliptic operator of order m on a compact smooth manifold with boundary, does the boundary image of its even relative K-homology class possess an ordinary L^2-Fredholm representative obtained from the complete Cauchy trace? The answer is positive. The representative is the projection onto $\widehat{H}_D = \lambda_{1/2}^{-1}\gamma(\operatorname{Ker} D_{\max})$, and its symbol is the Calderon bundle $E_+(D)$ of inward-decaying initial values.

This classification of domains is the basis for the paper. The minimal graph is taken from smooth functions with compact support and does not possess the entire boundary value information, whereas the maximal graph connects to the boundary via distributional L^2 solutions. The boundary class emerges out of the difference, rather than any external prescription.

2. Operator Objects and Boundary Transfer

Let M be a smooth manifold with a fixed smooth density and let $E, F \to M$ be Hermitian vector bundles. For

$$D \in \operatorname{Diff}(M; E, F),$$

the formal adjoint $D^\dagger \in \operatorname{Diff}(M; F, E)$ is defined by

$$\langle Df, g\rangle_{L^2(M;F)} = \langle f, D^\dagger g\rangle_{L^2(M;E)}$$

for compactly supported smooth sections. On an open set $\Omega \subset M$, the minimal realization is

$$\operatorname{Dom} D_{\min} = \overline{C_c^\infty(\Omega; E)}^{\|\cdot\|_D}, \qquad \|u\|_D^2 = \|u\|_{L^2}^2 + \|Du\|_{L^2}^2,$$

and the maximal realization is

$$\operatorname{Dom} D_{\max} = \{u \in L^2(\Omega; E) : Du \in L^2(\Omega; F) \text{ distributionally}\}.$$

The adjoint relation

$$(D_{\max})^* = (D^\dagger)_{\min}$$

places $D_{\min}$ and $D_{\max}$ at opposite ends of the realization interval. Any closed realization acting distributionally as D lies between them.

For a non-self-adjoint operator the even relative object is

$$\mathcal{D}_D = \begin{pmatrix} 0 & (D^\dagger)_{\min} \\ D_{\min} & 0 \end{pmatrix}$$

on $L^2(\Omega; E \oplus F)$. The bounded transform of $\mathcal{D}_D$ defines a relative K-homology class once the higher-order compactness and commutator conditions hold. Locality supplies these conditions for elliptic differential operators restricted to an interior ideal. If $j \in C_c^\infty(\Omega)$, a cut-off $k \in C_c^\infty(\Omega)$ with $k = 1$ near $\operatorname{supp} j$ gives

$$j = kj, \qquad Dj = kDj.$$

Thus the differential action remains inside the ideal generated by compactly supported functions.

Table 1: Boundary objects.

Object	Definition	Role
Minimal graph	$\operatorname{Dom} D_{\min} = \overline{C_c^\infty(\Omega; E)}^{\lVert\cdot\rVert_D}$	Carries the interior relative operator.
Maximal graph	$\operatorname{Dom} D_{\max} = \{u \in L^2 : Du \in L^2\}$	Contains boundary-visible solutions.
Even operator	$\mathcal{D}_D = \begin{pmatrix} 0 & (D^\dagger)_{\min} \\ D_{\min} & 0 \end{pmatrix}$	Produces the relative K-class.
Bergman space	$\operatorname{Ker} D_{\max}$	Represents homogeneous maximal solutions.
Cauchy trace	$\gamma u = (\gamma_0 u, \ldots, \gamma_{m-1} u)$	Retains all normal channels.
Normalized range	$\widehat{H}_D = \lambda_{1/2}^{-1} \gamma(\operatorname{Ker} D_{\max})$	Gives an ordinary boundary Hilbert subspace.
Boundary projection	$\Pi_D = P_{\widehat{H}_D}$	Defines the odd Fredholm module.
Calderon bundle	$E_+(D) \to S^*\partial\Omega$	Provides the topological symbol.

These are all the mathematical objects needed in the paper; they depend on the operator, adjoint, graph norm, trace theorem, and principal boundary symbol. In addition, the entries reveal how m is incorporated: it is present in the number of trace channels and the boundary space $E \otimes \mathbb{C}^m$.

The transition from D to $[D, a]$ reduces the differential order by 1, while the fractional resolvent power provides just enough extra regularity to get a bounded operator on the Hilbert space.

The equalities $j = kj$ and $Dj = kDj$ provide the concrete reason behind the relative ideal property: local multiplication cannot push interior data supported in a neighborhood to the boundary.

The bounded transformation leads to a stabilization of the unbounded relative operator; after localization, it becomes compact for the adjoint, square, and commutator.

3. Normal-Trace Compression

Assume now that Ω is compact with smooth boundary and that $D : C^\infty(\Omega; E) \to C^\infty(\Omega; F)$ is classically elliptic of order m. The full trace

$$\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{m-1})$$

extends to the maximal domain in the weak trace sense. In the smooth elliptic case, the trace kernel on $\operatorname{Dom} D_{\max}$ is the minimal domain. Hence

$$\operatorname{Ker} \gamma = \operatorname{Dom} D_{\min}.$$

This identity means that the quotient between maximal and minimal graph behavior is recorded by the Cauchy trace.

The homogeneous maximal solution space

$$\mathcal{N}_D = \operatorname{Ker} D_{\max}$$

forms the generalized Bergman space of D. Its boundary values give the Cauchy space

$$H_D = \gamma(\mathcal{N}_D).$$

The Calderon projector of Seeley has range H_D and Cauchy data pseudodifferential representation [12]. The characterization of Grubb gives description of the non-local elliptic boundary value problems associated [33]. Then, functional calculus accounts for the various order changes that occur in the trace part [34]. An ordinary L^2-projection is obtained only after order alignment.

Let Λ_β be a positive elliptic operator of order β on $\partial\Omega$, and set

$$\lambda_{1/2} = \operatorname{diag}(\Lambda_{1/2}, \Lambda_{3/2}, \ldots, \Lambda_{m-1/2}).$$

The normalized Cauchy space is

$$\widehat{H}_D = \lambda_{1/2}^{-1} H_D \subset L^2(\partial\Omega; E \otimes \mathbb{C}^m).$$

The orthogonal projection $\Pi_D = P_{\widehat{H}_D}$ is an order-zero pseudodifferential projection modulo compact operators. Therefore

$$(C(\partial\Omega), L^2(\partial\Omega; E \otimes \mathbb{C}^m), 2\Pi_D - 1)$$

is an odd Fredholm module.

The connecting map for

$$0 \longrightarrow C_0(\Omega) \longrightarrow C(\overline{\Omega}) \longrightarrow C(\partial\Omega) \longrightarrow 0.$$

This precise chain is an encoding of the boundary obstruction. A non-trivial boundary class prohibits the relative cycle from lifting to an arbitrary absolute class on $C(\overline{\Omega})$.

The map from $\operatorname{Ker} D_{\max}$ to H_D keeps the boundary term of the homogeneous maximal solutions via all traces, and not only via the zeroth one.

The operator $\lambda_{1/2}^{-1}$ does not compress the different Sobolev layers; it pushes them into a unified L^2 layer, keeping the number of boundary paths unchanged.

The chain of verifications is schematically listed in Table 2. The table also ensures that the formulae will not stand by themselves: each identity has its analytical task.

Table 2: Analytical checks.

Step	Operation	Required identity
Relative class	Bound $\mathcal{D}_D$ by functional calculus.	Compact localized defects for the transform.
Boundary map	Apply the connecting homomorphism.	$\partial[\mathcal{D}_D] = [P_{\operatorname{Ker} D_{\max}}]$.
Trace passage	Send maximal solutions to Cauchy data.	$H_D = \gamma(\operatorname{Ker} D_{\max})$.
Order alignment	Apply $\lambda_{1/2}^{-1}$.	$\widehat{H}_D \subset L^2(\partial\Omega; E \otimes \mathbb{C}^m)$.
Projection class	Use $\Pi_D = P_{\widehat{H}_D}$.	$[\Pi_D] = [P_{\operatorname{Ker} D_{\max}}]$.
Symbol reading	Freeze tangential variables at $S^*\partial\Omega$.	$\sigma_0(\Pi_D) = [E_+(D)]$.

These are the exact identities that are required for the boundary representative. None of them requires analogy to the first-order Dirac operator. The order of D can be seen through the trace and symbol levels.

4. Boundary Symbol and Fredholm Pairing

The leading symbol of the projection is found from the ordinary differential equation along the inward normal direction. In product coordinates (x', x_n) near $\partial\Omega$, freeze the tangential covector $(x', \xi') \in S^*\partial\Omega$ and form

$$\sigma_\partial(D)(x', \xi')v = 0.$$

The initial values

$$(v(0), D_t v(0), \ldots, D_t^{m-1} v(0))$$

of the solutions decaying as $t \to +\infty$ form the vector bundle $E_+(D) \to S^*\partial\Omega$. The order-zero principal symbol of Π_D is the projection onto this bundle.

The inward half-line in the boundary ordinary differential equation selects the decaying branch. Its initial values determine $E_+(D)$, the topological symbol carried by Π_D.

For $\alpha \in C(\partial\Omega; \operatorname{End}(\mathbb{C}^N))$ invertible, multiplication by α gives a compressed operator

$$T_D(\alpha) = \Pi_D \alpha \Pi_D$$

on $\widehat{H}_D^N$. Since $[\Pi_D, \alpha]$ is compact, $T_D(\alpha)$ is Fredholm. The index is the pairing

$$\langle [\alpha], [\Pi_D] \rangle.$$

The symbol formula is

$$\operatorname{ind} T_D(\alpha) = \int_{S^*\partial\Omega} \operatorname{ch}(\varphi^*(\alpha)) \wedge \operatorname{ch}(E_+(D)) \wedge \operatorname{Td}(S^*\partial\Omega),$$

where $\varphi : S^*\partial\Omega \to \partial\Omega$ is the cosphere projection.

Reduction of the boundary multiplication map to the normalized Cauchy space results in a Fredholm map whose index captures the pairing of the invertible boundary coefficient and the odd K-homology class represented by Π_D.

5. Results and Interpretation

The main result is the identity

$$\partial[\mathcal{D}_D] = [\Pi_D] \quad \text{in} \quad K_1(C(\partial\Omega)).$$

The proof is based on the generalized Bergman representation of the boundary operator and its transportation to the Cauchy trace. The Poisson operator allows for identification of Cauchy data with homogeneous maximal solutions up to finite- and compact-dimensional terms. The change of order does not affect the K-class of the Cauchy space, although it alters its Hilbert representation. Consequently, the normalized projection defines the same odd boundary K-class as the operator $P_{\operatorname{Ker} D_{\max}}$.

The second result is the symbol relation

$$[\Pi_D] = \varphi_*\big([E_+(D)] \cap [S^*\partial\Omega]\big).$$

This relation connects the projection onto $L^2(\partial\Omega; E \otimes \mathbb{C}^m)$ with the decaying initial value bundle of the boundary normal operator. This is precisely the point where the Calderon calculus becomes a K-homology statement.

The third result is about extensions. Whenever

$$D_{\min} \subseteq D_e \subseteq D_{\max}$$

is satisfied for a given realization D_e, the bounded transform corresponding to each realization gives the same interior K-homology class. Therefore, a boundary condition only modifies the representative and not the relative class. A vanishing boundary image is needed to have an extension into the full boundary algebra.

The fourth result is about lower-order stability. The replacement of D by $D + R$, where R has order strictly less than m, keeps the K-homology class intact provided that the leading boundary symbol is preserved. Lower-order perturbations modify the kernel and finite-frequency information, but they do not affect $[E_+(D)]$ or $[\Pi_D]$.

The fifth result is self-adjoint cancellation. If the operator D is self-adjoint and satisfies the closed range property, the scalar Toeplitz pairings vanish even though the Cauchy projection is non-trivial. This is due to the vanishing equality

$$\operatorname{ind}(P_{\operatorname{Ker} D_{\max}} \alpha P_{\operatorname{Ker} D_{\max}}) = 0$$

for invertible scalar boundary coefficients.

Table 3: Boundary consequences.

Conclusion	Mathematical statement	Interpretation
Boundary representation	$\partial[\mathcal{D}_D] = [\Pi_D]$	The normalized Cauchy projection is the boundary class.
Symbol retention	$[\Pi_D] = \varphi_*([E_+(D)] \cap [S^*\partial\Omega])$	The Calderon symbol survives order alignment.
Extension invariance	$[D_e] = [D_{\min}]$ over the interior ideal	Boundary conditions do not change the relative class.
Perturbation stability	$[D + R] = [D]$ for lower-order R	The leading boundary symbol controls the class.
Self-adjoint cancellation	Scalar pairings vanish under formal self-adjointness	A zero integer pairing can coexist with non-zero Cauchy data.

The conclusions reveal why normal-trace compression is a more powerful tool than a theorem on the existence of Calderon projectors in terms of giving one Hilbert-space representative, one boundary K-class, and one symbol identity with the higher-order trace multiplicity preserved.

These invariance features have a uniform interpretation. Realizations between $D_{\min}$ and $D_{\max}$, perturbations of a lower order, and self-adjoint cancelations impact the representatives or pairings, but K-homology interpretation is always fixed by the leading-order boundary symbol and the normalized Cauchy projection.

6. Discussion

It should be noticed that the definition of the boundary class in terms of the normalized trace distinguishes it from the case of a single trace or a realization choice. This is important as the order of the operator is revealed at the boundary after the normal traces are retained. First-order Dirac-type boundary problems give the standard example of this construction [35]. Elliptic boundary problems and the Cauchy data for such systems have been worked out in the operator-theoretic language as well [36]. A first-order Dirac operator has only one boundary channel corresponding to the boundary spectral subspace. An order-m operator has m channels which differ in their Sobolev orders. The diagonal operator $\lambda_{1/2}$ preserves this information and expresses it via $L^2(\partial\Omega; E \otimes \mathbb{C}^m)$ where Fredholm modules over $C(\partial\Omega)$ are considered in the usual manner.

Now the connection with the Baum-Douglas-Taylor identity is clear. Spin geometry provides the Clifford-module and Dirac-operator setup for the first-order case [37]. In the case of a spin^c Dirac operator, $E_+(D)$ is the positive boundary spectral bundle, and the boundary class becomes simply the fundamental class of $\partial\Omega$. In this sense,

$$\partial[\Omega] = [\partial\Omega]$$

is obtained as the first-order spin^c example [16]. For higher-order operators, $E_+(D)$ is defined by an m-th order normal differential equation, and the trace vector must be taken into account.

The connection with Boutet de Monvel's index theorem is also immediate. In the latter setting, the Fredholm operator $T_D(\alpha) = \Pi_D \alpha \Pi_D$ is a boundary Toeplitz compression whose index is given by the Chern character of α, the Calderon bundle, and the Todd class of the cosphere bundle. The formula does not require any special boundary condition since it uses the boundary class carried by the maximal homogeneous solutions.

The extension result leads to an obstruction interpretation. Realizing D means finding a boundary condition within the range of the interval of minimal and maximal domains. For such a realization, an absolute K-cycle over $C(\overline{\Omega})$ would mean that the boundary class vanishes. Thus, a non-vanishing $[\Pi_D]$ means something specific – there is no lifting the relative class without running into the boundary obstruction.

Low order terms are analytically present but invisible to the K-class if the high order symbol is fixed. Curvature, connection, potential, and tangential low order contributions might affect the kernels and the solutions in finite dimensions. The high normal order symbol alone determines $E_+(D)$, and compact perturbation keeps the K-homology class intact. Normal-trace compression separates the contributions effectively – the projection representative can be changed, but its K-homology class will stay put.

The self-adjoint setting must not be misinterpreted. Zero scalar pairing does not imply zero boundary projection. Formally self-adjoint polyharmonic and other operators can have large trace spaces, while symmetry can destroy the integer scalar index. Still, the projection Π_D captures the Cauchy data and richer pairings or family index might reveal information hidden from the scalar one.

The limitations of the analysis are clear too. The most restrictive symbol statement requires compact smooth boundary and classical ellipticity. Rougher boundaries call for alternative trace theorems. Heisenberg manifold analysis allows for a non-classical calculus with anisotropic differential order [38]. Its spectral theory shows how hypoelliptic operators require adapted Sobolev and symbol scales [39]. K-homology on contact manifolds proves that index classes are robust against ellipticity [40]. Groupoid techniques provide a uniform construction of such pseudodifferential calculi [41]. While the interior relative cycle construction is applicable in this context, the boundary projection and order reduction must be adjusted accordingly. In the current work, only trace scale, Calderon idempotent, and order reduction are isolated as such.

7. Conclusion

There is indeed an answer to the question posed in the introduction. The boundary K-theory class of the even relative class of a classically elliptic order-m operator is definable using one ordinary boundary Hilbert space, but only when the complete Cauchy trace is used and normalized. The representative is

$$\Pi_D = P_{\widehat{H}_D}, \qquad \widehat{H}_D = \lambda_{1/2}^{-1}\gamma(\operatorname{Ker} D_{\max}) \subset L^2(\partial\Omega; E \otimes \mathbb{C}^m).$$

One finds

$$\partial[\mathcal{D}_D] = [\Pi_D],$$

while the principal symbol gives

$$[\Pi_D] = \varphi_*\big([E_+(D)] \cap [S^*\partial\Omega]\big).$$

Consequently, normal-trace compression is precisely the operator-theoretic process connecting minimal-domain relative K-homology, generalized Bergman solutions, Calderon Cauchy data, and the topological boundary symbol.

The conclusion is not merely that higher-order differential operators have boundary value problems. It is that the entire order-m boundary vector must first be compressed down to $L^2(\partial\Omega; E \otimes \mathbb{C}^m)$, whereupon it becomes an ordinary Fredholm module representing a boundary K-class. This compression retains the Calderon symbol, is independent of the choice of extension, is invariant under lower order perturbations, and accounts for the fact that formal self-adjointness may kill integer pairs without destroying the boundary Cauchy structure.

Conflict of Interest

The author declares no conflict of interest.

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